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Unified derivation of Korteweg–de Vries–Zakharov–Kuznetsov equations in multispecies plasmas

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Abstract

Ordinary, modified and mixed Korteweg–de Vries–Zakharov–Kuznetsov (KdV–ZK) equations governing the oblique propagation of electrostatic modes in magnetized plasmas have been (re)derived in a fully systematic way for general mixtures of hot isothermal, warm adiabatic fluid and cold immobile background species. The ordinary KdV–ZK equation is the standard paradigm, but for more complicated plasma compositions the soliton character can switch from compressive to rarefactive or vice versa, at critical densities and temperatures. For these special values the modified KdV–ZK equation is to be used, whereas near such critical values a mixed KdV–ZK equation can model double layers. Since the description is given in physical rather than normalized variables for genuine multispecies plasmas, widely different frequency regimes and plasma models can be treated and the general features compared. Applications include electron- and ion-acoustic modes in normal plasmas with one or two hot Boltzmann electron species, or ion- and dust-acoustic modes in dusty plasmas, depending on how the heavier components are modelled. Special emphasis is given to a discussion of critical regimes for the better known plasma electrostatic modes, leading to new results and better physical insight.

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1. Introduction

Korteweg–de Vries–Zakharov–Kuznetsov (KdV–ZK) equations have been derived for quite a number of different electrostatic plasma modes that propagate obliquely to a static magnetic

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field $\boldsymbol{B}_0$. The simplest model is that of ion-acoustic solitons, where the hot isothermal electrons are described by a Boltzmann distribution, while the cooler ions are treated as a fluid with adiabatic pressures. To cite a few applications in plasma physics of KdV-like equations, derivations leading to the KdV equation itself have been given for parallel electrostatic (Washimi and Taniuti 1966), oblique electromagnetic (Kakutani *et al* 1968) and perpendicular magnetosonic modes (Kakutani and Ono 1969).

Many variations on this basic theme crop up in the literature, which is so vast that we can only cite a few, immediately relevant papers. One is where different positive and negative ions are present, so that a large enough concentration of negative ions can alter the character of the solitons from compressive to rarefactive (Das and Tagare 1975). To avoid ambiguities, we shall use the words compressive and rarefactive to refer to the behaviour of the electrostatic potential rather than to densities of particular species. The transition occurs at critical densities, for which the KdV–ZK equation is no longer the appropriate nonlinear paradigm, but one finds instead the modified KdV–ZK (mKdV–ZK) equation, with cubic rather than quadratic nonlinearities. Close to critical densities, mixed KdV–mKdV equations with both quadratic and cubic nonlinearities can occur, and describe solitons or weak double layers in plasmas (Raadu 1989).

Another possibility is to consider two distinct electron species, one hot and isothermal, the other cooler and adiabatic, which together with magnetized or unmagnetized ions leads to electron-acoustic solitons (Mace and Hellberg 2001). More recently, attention has turned to dusty plasmas, where one or more charged dust components have introduced space and timescales that differ vastly from those associated with the usual ions and electrons (Mendis and Rosenberg 1994, Bliokh *et al* 1995, Horányi 1996, Verheest 1996, Verheest 2000, Shukla 2001). Here the prime example is the dust-acoustic mode, well studied both in theory (Rao *et al* 1990) and in the laboratory (Barkan *et al* 1995).

Earlier treatments are too often restricted to specific models and use corresponding normalizations that make comparisons between analogous expressions for the different coefficients difficult. It is thus of interest to revisit the whole field of electrostatic modes in magnetized plasmas in a general treatment that encompasses all the known dispersion laws and nonlinear equations. For this, we shall distinguish three classes of species, each initially including an unspecified number of constituents. The cooler adiabatic species are described by standard fluid equations, including that governing the pressure variations. Besides these, we shall allow for two extremes: on the hot side, several Boltzmann species can be considered, corresponding effectively to the massless limit where inertial effects vanish because the thermal velocities exceed typical wave and translational speeds. On the sluggish side, several immobile background species are included, to simulate the case of unmagnetized ions when describing electron-acoustic solitons (Mace and Hellberg 2001), or of unmagnetized dust when dealing with ion-acoustic solitons in dusty plasmas. The latter are also called dust-ion-acoustic modes.

The paper is structured as follows. In section 2 we recall some elements of the basic formalism, and derive then in section 3 the generic nonlinear modes, governed by a KdV–ZK equation. For critical densities the nonlinear modes obey a mKdV–ZK equation, given in section 4. Although we include possible beam velocities in the general derivations, we shall leave a discussion of beam instabilities for a companion paper. Several specific applications then follow in section 5, including electron-acoustic, ion-acoustic and dust-acoustic solitons. Critical and supercritical densities are discussed where relevant. Finally, our conclusions are summarized in section 6.

2. Basic formalism

As pointed out in the introduction, the basic model includes three classes of species. The cooler adiabatic species, with running subscript α, are described by standard fluid equations. These are the continuity equations,

$$\frac{\partial n_\alpha}{\partial t} + \nabla \cdot (n_\alpha \boldsymbol{u}_\alpha) = 0 \tag{1}$$

the equations of motion,

$$\frac{\partial \boldsymbol{u}_\alpha}{\partial t} + \boldsymbol{u}_\alpha \cdot \nabla \boldsymbol{u}_\alpha + \frac{1}{n_\alpha m_\alpha} \nabla p_\alpha = -\frac{q_\alpha}{m_\alpha} \nabla \varphi + \Omega_\alpha \boldsymbol{u}_\alpha \times \boldsymbol{e}_x \tag{2}$$

and the adiabatic pressure equations,

$$\frac{\partial p_\alpha}{\partial t} + \boldsymbol{u}_\alpha \cdot \nabla p_\alpha + \gamma_\alpha p_\alpha \nabla \cdot \boldsymbol{u}_\alpha = 0. \tag{3}$$

Here n_α, $\boldsymbol{u}_\alpha$ and p_α refer to the densities, fluid velocities and pressures, respectively, of the different species. The latter have charges q_α, masses m_α, adiabatic indices γ_α and gyrofrequencies $\Omega_\alpha = q_\alpha B_0 / m_\alpha$ that include the sign of the charge. The direction of the static magnetic field $\boldsymbol{B}_0$ has been taken as the x axis of the reference frame. In this study of electrostatic modes φ denotes the electrostatic potential, and wave magnetic fields will be omitted.

In addition to the fluid species, there are a number of hot isothermal Boltzmann species, denoted by a running subscript β, with densities

$$n_\beta = N_\beta \exp\left[-\frac{q_\beta \varphi}{\kappa T_\beta}\right]. \tag{4}$$

Here T_β refers to the isothermal temperatures. For all species equilibrium quantities will be denoted by capital letters, such as N_β for the equilibrium densities of the Boltzmann species or U_α for the parallel equilibrium streaming of the adiabatic species, in order to avoid dealing with too many subscripts later on when expanding to different orders.

On the sluggish side, several immobile background species, with running subscript δ, have constant densities N_δ, and these are included to allow for (charge) density imbalances between the species responding to the waves.

Finally, the set of equations is closed by Poisson's equation

$$\varepsilon_0 \nabla^2 \varphi + \sum_\alpha n_\alpha q_\alpha + \sum_\beta N_\beta q_\beta \exp\left[-\frac{q_\beta \varphi}{\kappa T_\beta}\right] + \sum_\delta N_\delta q_\delta = 0. \tag{5}$$

Before addressing the nonlinear evolution, we briefly discuss the dispersion law for linear modes described by the set (1)–(5). This dispersion law is found to be

$$\sum_\alpha \omega_{p\alpha}^2 \frac{k^2 \hat{\omega}_\alpha^2 - k_\parallel^2 \Omega_\alpha^2}{\hat{\omega}_\alpha^4 - \hat{\omega}_\alpha^2 (k^2 v_{T\alpha}^2 + \Omega_\alpha^2) + k_\parallel^2 v_{T\alpha}^2 \Omega_\alpha^2} = k^2 + \sum_\beta \frac{1}{\lambda_{\mathrm{D}\beta}^2}. \tag{6}$$

Plasma frequencies $\omega_{\mathrm{p}\alpha}$ are defined through $\omega_{p\alpha}^2 = N_\alpha q_\alpha^2 / \varepsilon_0 m_\alpha$, Debye lengths $\lambda_{\mathrm{D}\beta}$ for the isothermal species through $\lambda_{\mathrm{D}\beta}^2 = \varepsilon_0 \kappa T_\beta / N_\beta q_\beta^2$ and Doppler-shifted wave frequencies as $\hat{\omega}_\alpha = \omega - k_\parallel U_\alpha$, with $k_\parallel$ the wavenumber parallel to the direction of the static magnetic field. Thermal velocities $v_{T\alpha}$ for the adiabatic species will be defined through $v_{T\alpha}^2 = \gamma_\alpha P_\alpha / N_\alpha m_\alpha$, with the inclusion of the adiabatic index γ_α. This explicit departure from the conventional definition is purely intended to lighten the notation. At parallel propagation, (6) reduces to

$$\sum_\alpha \frac{\omega_{p\alpha}^2}{\hat{\omega}_\alpha^2 - k^2 v_{T\alpha}^2} = 1 + \sum_\beta \frac{1}{k^2 \lambda_{\mathrm{D}\beta}^2}. \tag{7}$$

For small wavenumbers both dispersion laws (6) and (7) can be approximated to the lowest order as an acoustic-like dispersion-free propagation along the field, with a dispersive correction of order k^3, i.e.

$$\omega = k_\parallel V - k_\parallel^3 a - k_\parallel k_\perp^2 d + \cdots . \tag{8}$$

The phase velocity V in the limit of vanishing wavenumbers is determined from

$$\mathcal{D} \equiv \sum_\alpha \frac{\omega_{p\alpha}^2}{(V - U_\alpha)^2 - v_{T\alpha}^2} - \sum_\beta \frac{1}{\lambda_{\mathrm{D}\beta}^2} = 0 \tag{9}$$

and the coefficients a and d are given by

$$a = \frac{1}{A} \qquad d = \frac{D}{A} \tag{10}$$

with

$$A = 2 \sum_\alpha \frac{\omega_{p\alpha}^2 (V - U_\alpha)}{[(V - U_\alpha)^2 - v_{T\alpha}^2]^2} \tag{11}$$

$$D = 1 + \sum_\alpha \frac{\omega_{p\alpha}^2 (V - U_\alpha)^4}{\Omega_\alpha^2 [(V - U_\alpha)^2 - v_{T\alpha}^2]^2} . \tag{12}$$

As we shall see afterwards, a and d will turn out to be the coefficients of the dispersive terms in the KdV–ZK equation (19) and the mKdV–ZK equation (24). A physically more transparent discussion of (8) will be given in the subsection dealing with the application of our general formalism to electron-acoustic waves.

The immobile species play no direct role in the dispersion laws (6) and (9), but have an indirect effect through the Poisson equation on n_α and on N_β, and hence on $\omega_{\mathrm{p}\alpha}$ and on $\lambda_{\mathrm{D}\beta}$. This is not surprising as it is understandable that an immobile species would play no direct role in wave motion.

3. Generic nonlinear modes: KdV–ZK equation

Inspired by the properties of the linear dispersion law for small wavenumbers k, we adopt the standard KdV stretching of the independent variables,

$$\begin{aligned} \xi &= \varepsilon^{1/2}(x - Vt) \\ \eta &= \varepsilon^{1/2} y \\ \zeta &= \varepsilon^{1/2} z \\ \tau &= \varepsilon^{3/2} t . \end{aligned} \tag{13}$$

Here V is the velocity of the nonlinear structure in a co-moving frame, when nonlinearities and dispersion are omitted. Thus, V basically corresponds to the linear phase velocity in the limit $k \to 0$, and will later be shown in a natural way to obey (9).

To be fully systematic and allow for the derivations of both the KdV–ZK and mKdV–ZK equations in a coherent way, we adopt a general expansion for all dependent variables in powers of $\varepsilon^{1/2}$, of the form

$$f = F + \varepsilon^{1/2} f_1 + \varepsilon f_2 + \varepsilon^{3/2} f_3 + \varepsilon^2 f_4 + \cdots . \tag{14}$$

Only the densities, pressures and parallel components of the fluid velocities will have nonzero equilibrium values, denoted respectively by N_α, P_α and U_α. The latter then give the possibility of including parallel beam effects and instabilities. The perpendicular components of the fluid velocities and the electrostatic potential are assumed to vanish in equilibrium.

Inserting the stretching (13) and the expansions (14) into the basic equations (1)–(5) gives a series of equations, upon separating out the different orders in $\varepsilon^{1/2}$. When possible, these equations will be integrated with one-sided boundary conditions suitable for the type of solitary wave structure we are interested in,

$$f \to F \qquad \text{and} \qquad f_{\rm i} \to 0 \qquad \frac{\partial f_{\rm i}}{\partial \xi} \to 0 \qquad (i = 1, 2, \ldots) \tag{15}$$

when $\xi \to \infty$. This procedure is standard and the intermediate steps are well documented in the literature for some of the simpler cases (Das and Verheest 1989, Das *et al* 1992, Murawski and Edwin 1992, Mishra *et al* 1994, Verheest and Hellberg 1999, Das *et al* 2000, Mamun *et al* 2000), so that there is no need to spell out the intervening results, since our computations are analogous and straightforward. We shall only highlight what is obtained from the Poisson equation (5) at various orders in $\varepsilon^{1/2}$. To order ε^0 we have overall charge neutrality in equilibrium,

$$\sum_\alpha N_\alpha q_\alpha + \sum_\beta N_\beta q_\beta + \sum_\delta N_\delta q_\delta = 0 \tag{16}$$

whereas to order $\varepsilon^{1/2}$ the dispersion law (9) determining V is recovered. Contrary to what is customarily done in the literature, we have deliberately not used non-dimensional or normalized variables, because one of the aims of our paper is to look at widely varying plasma compositions and wave domains, for which common normalizations are neither possible nor meaningful.

Next, to order ε, a natural bifurcation point is reached, namely

$$\mathcal{D}\varphi_2 + B\varphi_1^2 = 0. \tag{17}$$

The terms in φ_2 drop out, on account of the linear dispersion law (9), and B is given by

$$B = \sum_\alpha \frac{\omega_{p\alpha}^2 q_\alpha [3(V - U_\alpha)^2 + v_{T\alpha}^2]}{m_\alpha[(V - U_\alpha)^2 - v_{T\alpha}^2]^3} + \sum_\beta \frac{q_\beta}{\lambda_{{\rm D}\beta}^2 \kappa T_\beta}. \tag{18}$$

It will be shown that this is essentially the coefficient of the quadratic nonlinearity in a KdV–ZK equation.

Although for electron–proton plasmas B is strictly positive, for certain more complicated plasma compositions the parameters are such that B can become negative (Das and Tagare 1975, Verheest 1988) and there exist critical densities obeying $B = 0$. Because $B\varphi_1^2 = 0$, the bifurcation means that either the plasma composition is very special, so that $B = 0$, to be discussed further on, or $\varphi_1 = 0$ and all variables with subscript 1 vanish from the expansions (14).

We proceed with the generic case where $B \neq 0$, and hence $\varphi_1 = 0$, indicating that there are no terms of order $\varepsilon^{1/2}$ in the expansions (14) of the dependent variables. This is the normal KdV expansion, except that the lowest-order variables now carry a subscript 2 (instead of 1 as usual in the literature) because of our treatment of the KdV–ZK and mKdV–ZK equations in one coherent derivation.

It turns out that to order $\varepsilon^{3/2}$ Poisson's equation (5) merely duplicates for φ_3 what we learned to order ε for φ_2. On the other hand, new information is obtained to order ε^2, and this gives the KdV–ZK equation (Zakharov and Kuznetsov 1974, Laedke and Spatschek 1982, Infeld 1985, Das and Verheest 1989, Infeld and Frycz 1991, Edwin and Murawski 1995, Mamun 1998, Infeld and Rowlands 2000)

$$\frac{\partial \varphi_2}{\partial \tau} + a\frac{\partial^3 \varphi_2}{\partial \xi^3} + b\varphi_2 \frac{\partial \varphi_2}{\partial \xi} + d\frac{\partial}{\partial \xi}\left(\frac{\partial^2 \varphi_2}{\partial \eta^2} + \frac{\partial^2 \varphi_2}{\partial \zeta^2}\right) = 0 \tag{19}$$

where the new coefficient b is given by

$$b = \frac{B}{A} \tag{20}$$

and a and d have already been defined in (10) when discussing cubic corrections to the lowest-order linear dispersion law. The expression for parallel propagation is recovered by omitting all η and ζ dependences, giving instead of (19) a standard KdV equation, formally obtained as if $d = 0$. Similar remarks will be made in the next section when encountering the mKdV–ZK equation (24), which reduces for parallel propagation to the corresponding mKdV equation.

It is clear that the slow time variations are caused only by the adiabatic fluid constituents of the plasma, through A, whereas the oblique effects, through D, contain a part which comes from the Laplacian and another that is related to the fluid species. The hot Boltzmann species only enter through B, the coefficient of the nonlinear term.

In principle A and B can change sign, whereas D is strictly positive. When one looks at the structure of the coefficients (10) and (20) in (19), it is clear that A comes from the slow time derivative, and the sign of A could, in principle, be absorbed by a time reversal, giving no new physical insight. The sign of A depends on the sign of the various quantities $V - U_\alpha$, and for a stationary plasma A is always positive. Since D is strictly positive, it is the possible sign change of B that leads to physically different situations. Indeed, if critical densities can be exceeded so that B is negative, the solitons will be rarefactive. The transition from compressive to rarefactive, of course, occurs at $B = 0$, except that in the immediate vicinity thereof the expansions break down and have to be reconsidered.

The standard one-soliton solution (Ablowitz and Clarkson 1991, Infeld and Rowlands 2000) of (19) propagating at an angle ϑ to the static magnetic field is given by

$$\varphi_2 = \frac{3M}{b\cos\vartheta}\,\mathrm{sech}^2\left[\frac{1}{2}\mu\,\Xi\right] \tag{21}$$

where M is the soliton velocity, μ a measure of the inverse width given through

$$\mu^2 = \frac{M}{(a\cos^2\vartheta + d\sin^2\vartheta)\cos\vartheta} \tag{22}$$

Ξ the running phase argument,

$$\Xi = \xi\cos\vartheta + \eta\sin\vartheta\cos\psi + \zeta\sin\vartheta\sin\psi - Mt \tag{23}$$

and ψ the second angle in spherical coordinates.

4. Nonlinear modes at critical densities: mKdV–ZK equation

Now we return to the bifurcation point encountered in the previous section and assume that the plasma is at critical densities, defined here by putting $B = 0$. This condition will be discussed in more detail in the next sections for different plasma compositions. Now $B = 0$ implies that we can continue to work with φ_1. The mKdV–ZK equation then follows from (5) to order $\varepsilon^{3/2}$, namely

$$\frac{\partial\varphi_1}{\partial\tau} + a\frac{\partial^3\varphi_1}{\partial\xi^3} + c\varphi_1^2\frac{\partial\varphi_1}{\partial\xi} + d\frac{\partial}{\partial\xi}\left(\frac{\partial^2\varphi_1}{\partial\eta^2} + \frac{\partial^2\varphi_1}{\partial\zeta^2}\right) = 0. \tag{24}$$

The coefficients a and d are unchanged from their expressions (10) for the KdV–ZK equation (19), whereas the coefficient of the new, cubic term is given by

$$c = \frac{C}{A} \tag{25}$$

with

$$C = \frac{3}{2}\sum_\alpha \frac{\omega_{p\alpha}^2 q_\alpha^2[5(V-U_\alpha)^4 + 10(V-U_\alpha)^2 v_{T\alpha}^2 + v_{T\alpha}^4]}{m_\alpha^2[(V-U_\alpha)^2 - v_{T\alpha}^2]^5} - \frac{1}{2}\sum_\beta \frac{q_\beta^2}{\lambda_{\mathrm{D}\beta}^2\kappa^2 T_\beta^2}. \tag{26}$$

We note that the coefficient of the cubic nonlinearity C depends on both the adiabatic and the inertialess components, as was the case for B, the coefficient of the quadratic nonlinearity in the usual KdV–ZK equation (19). A change of sign of C (and hence the transition through $C = 0$) might be possible, depending on the relative balance between the contributions of the Boltzmann and the fluid species.

The one-soliton solution (Ablowitz and Clarkson 1991, Infeld and Rowlands 2000) of (24) propagating at an angle ϑ to the static magnetic field is given by

$$\varphi_1 = \pm\sqrt{\frac{6M}{c\cos\vartheta}}\operatorname{sech}\mu\,\Xi \tag{27}$$

and the parameters have been defined in (22) and (23).

In the vicinity of critical densities double layers become possible. For these to occur, one would need that $B\phi_1^2$ become small, of the order of $C\phi_1^3$, so that both quadratic and cubic nonlinearities can together be present in one evolution equation, the mixed KdV–ZK equation,

$$\frac{\partial\varphi_1}{\partial\tau} + a\frac{\partial^3\varphi_1}{\partial\xi^3} + b\varphi_1\frac{\partial\varphi_1}{\partial\xi} + c\varphi_1^2\frac{\partial\varphi_1}{\partial\xi} + d\frac{\partial}{\partial\xi}\left(\frac{\partial^2\varphi_1}{\partial\eta^2} + \frac{\partial^2\varphi_1}{\partial\zeta^2}\right) = 0. \tag{28}$$

This has general travelling solitary wave solutions of the form

$$\varphi_1 = \frac{6M}{b\cos\vartheta}\,\frac{1}{1 \pm \sqrt{1 + \frac{6Mc}{b^2\cos\vartheta}}\cosh\mu\,\Xi} \tag{29}$$

with μ and Ξ defined as before.

Weak double layers are possible if $6Mc + b^2\cos\vartheta = 0$, and are of the form

$$\varphi_1 = \pm\frac{3M}{b\cos\vartheta}\left(1 - \tanh\frac{1}{2}\mu\,\Xi\right). \tag{30}$$

However, the existence of weak double layers involves some tricky discussions about the validity of the expansions assumed in the singular perturbation scheme (Hellberg *et al* 1992) and will not be pursued here.

5. Applications

Among the many possible applications, we just highlight some relevant results and new insights.

5.1. *Weakly nonlinear electron-acoustic waves*

Electron-acoustic waves occur in plasmas having two electron components distinguished by their disparate temperatures. The simplest possible plasma model that supports the electron-acoustic wave is therefore a stationary plasma ($U_\alpha = 0$) which has protons, cool electrons with temperature T_c and density N_c and hot electrons with temperature T_h and density N_h. Within the context of the model outlined in section 2, this means that the cool electrons are fluid-like, obeying the fluid equations (1)–(3) (they are an α-component), the hot electrons, owing to their much greater mobility, may be treated as inertialess and hence obey Boltzmann's relation (they are a β-component) and the ions, which are stationary on electron-acoustic timescales, are a charge-neutralizing δ-component. We shall denote cool-electron parameters with subscript c, hot-electron parameters with subscript h and ions with subscript i.

With the above plasma model equation (9) gives the following for V^2:

$$V^2 = v_{\text{ea}}^2 + v_{Tc}^2 \tag{31}$$

where the electron-acoustic speed is defined by $v_{\rm ea} = \omega_{\rm pc}\lambda_{\rm Dh} = (N_{\rm c}/N_{\rm h})^{1/2}(\kappa T_{\rm h}/m_{\rm e})^{1/2}$, and $v_{T{\rm c}}^2 = \gamma_{\rm c}\kappa T_{\rm c}/m_{\rm e}$.

The KdV–ZK coefficients (10) and (20) are

$$a = \frac{1}{2}\frac{v_{\rm ea}^2\lambda_{\rm Dh}^2}{(v_{\rm ea}^2 + v_{T{\rm c}}^2)^{1/2}} \tag{32}$$

$$b = -\frac{1}{2}\frac{e}{v_{\rm ea}^2(v_{\rm ea}^2 + v_{T{\rm c}}^2)^{1/2}}\left[\frac{3v_{\rm ea}^2 + 4v_{T{\rm c}}^2}{m_{\rm e}} + \frac{v_{\rm ea}^4}{\kappa T_{\rm h}}\right] \tag{33}$$

$$d = \frac{1}{2}\frac{v_{\rm ea}^2\lambda_{\rm Dh}^2}{(v_{\rm ea}^2 + v_{T{\rm c}}^2)^{1/2}} + \frac{1}{2}\rho_{\rm se}^2(v_{\rm ea}^2 + v_{T{\rm c}}^2)^{1/2} \tag{34}$$

where $\rho_{\rm se} \equiv (v_{\rm ea}^2 + v_{T{\rm c}}^2)^{1/2}/\Omega_{\rm e}$ is the Larmor radius of an electron travelling at the linear parallel phase (or group) speed. The coefficients (32)–(34) are equivalent to those derived in Mace and Hellberg (2001), here made physically transparent through lack of normalization.

The coefficients a and d, with $v_{T{\rm c}} = 0$, are readily identified from the linear dispersion relation in the small-wavenumber limit (see, e.g., Mace and Hellberg (1993a) and our discussion of (8) in its more general form)

$$\omega - k_\parallel v_{\rm ea} = -\tfrac{1}{2}k_\parallel^3\lambda_{\rm Dh}^2 v_{\rm ea} - \tfrac{1}{2}k_\parallel k_\perp^2(\rho_{\rm se}^2 + \lambda_{\rm Dh}^2)v_{\rm ea}. \tag{35}$$

To see this note that $\mathrm{i}k_\parallel \to \partial/\partial\xi$ and $-k_\perp^2 \to \partial^2/\partial\eta^2 + \partial^2/\partial\zeta^2$. Then, as one would expect, the coefficient a controls dispersion (in a frame of reference propagating at the group velocity $v_{\rm ea}$) along the field direction and d, which contains finite-Larmor-radius effects, controls dispersion perpendicular to it.

The nonlinear term, whose coefficient is b, produces the coupling between Fourier components with different wavenumbers, i.e. wave–wave interaction. In the simple stationary model it is always negative and as a result only negative potential solitary waves are permitted. Of course, its negative definiteness rules out higher-order nonlinear evolutionary equations for electron-acoustic waves, unless a beam is included in the model (Berthomier *et al* 2000, Mace and Hellberg 2001). Mace and Hellberg (1993b) have shown furthermore that even if one relaxes the criterion that the nonlinear coefficient vanish, and requires it merely to be of order ε, one cannot derive a modified KdV equation without violating convergence criteria.

Some mention should be made of the choice of the parameter $\gamma_{\rm c}$, the ratio of specific heats for the cool-electron fluid. For planar one-dimensional solitons or wavepackets a value of $\gamma_{\rm c} = 3$ would be appropriate, but for multi-dimensional solitons the value of $\gamma_{\rm c}$ should be chosen to be two in the case of cylindrical solitons or 5/3 for spherical solitons. The latter choice of $\gamma_{\rm c} = 5/3$ would be an improvement on the value of three used in the model for ellipsoidal solitons described by Mace and Hellberg (2001).

5.2. Ion-acoustic solitons in two-electron-temperature plasmas

Critical densities can occur when there are two Boltzmann species of the same sign, or when two fluid species have different signs. Indeed, for ion-acoustic solitons in a plasma with two Boltzmann electron species (one hotter species, with subscript h, and one cooler species, with subscript c) and singly charged fluid ions, we find, again in the absence of equilibrium streaming, that the dispersion law (9) gives

$$V^2 = \omega_{pi}^2\lambda_{\rm D}^2 + v_{T{\rm i}}^2 = v_{\rm ia}^2 + v_{T{\rm i}}^2 \tag{36}$$

where $v_{\rm ia} = \omega_{\rm pi}\lambda_{\rm D}$ is the ion-acoustic velocity in the two-temperature plasma. Here $\lambda_{\rm D}$ is a global Debye length, given through $\lambda_{\rm D}^{-2} = \lambda_{\rm Dh}^{-2} + \lambda_{\rm Dc}^{-2}$, and needed whenever more than one Boltzmann species is considered (Verheest and Hellberg 1997). In this plasma we have that

$$B = \frac{\varepsilon_0}{N_\mathrm{i} e}\left[\frac{4v_{T\mathrm{i}}^2}{v_\mathrm{ia}^2\lambda_\mathrm{D}^4} + \frac{2}{\lambda_\mathrm{D}^4} - \frac{1}{N_\mathrm{h}N_\mathrm{c}}\left(\frac{N_\mathrm{h}}{\lambda_\mathrm{Dc}^2} - \frac{N_\mathrm{c}}{\lambda_\mathrm{Dh}^2}\right)^2\right]. \tag{37}$$

This can become zero in the regime where $T_\mathrm{c} \ll T_\mathrm{h}$ and $N_\mathrm{c} \ll N_\mathrm{h}$, so $N_\mathrm{c}/N_\mathrm{h} \sim T_\mathrm{c}^2/T_\mathrm{h}^2$. Analogous results were discussed in the framework of arbitrary-amplitude theory at parallel propagation (Baboolal *et al* 1989, Baboolal *et al* 1990, where many references to earlier work can be found).

Even supercritical densities are possible in this model, in the sense that both B and C vanish at the same time, when the plasma parameters also obey $N_\mathrm{i} = N_\mathrm{h} + N_\mathrm{c}$ and the appropriate dispersion law. For simplicity we shall investigate this in the cold-ion limit, because this only amounts to small corrections, without invalidating the gist of the computations. Using equilibrium charge neutrality allows us to rewrite (9), (18) and (26) together as

$$\begin{aligned}
\left(1 - \frac{m_\mathrm{i}V^2}{\kappa T_\mathrm{h}}\right)N_\mathrm{h} &= \left(\frac{m_\mathrm{i}V^2}{\kappa T_\mathrm{c}} - 1\right)N_\mathrm{c}\\
\left(3 - \frac{m_\mathrm{i}^2V^4}{\kappa^2 T_\mathrm{h}^2}\right)N_\mathrm{h} &= \left(\frac{m_\mathrm{i}^2V^4}{\kappa^2 T_\mathrm{c}^2} - 3\right)N_\mathrm{c}\\
\left(15 - \frac{m_\mathrm{i}^3V^6}{\kappa^3 T_\mathrm{h}^3}\right)N_\mathrm{h} &= \left(\frac{m_\mathrm{i}^3V^6}{\kappa^3 T_\mathrm{c}^3} - 15\right)N_\mathrm{c}.
\end{aligned} \tag{38}$$

This leads to the following special relations between the temperatures and densities of the Boltzmann electron species:

$$\frac{N_\mathrm{c}}{N_\mathrm{h}} = \frac{T_\mathrm{c}}{T_\mathrm{h}} = 5 - 2\sqrt{6} \simeq 0.1 \tag{39}$$

with the phase velocity V given by

$$V^2 = (3 - \sqrt{6})\frac{\kappa T_\mathrm{h}}{m_\mathrm{i}} \simeq 0.55\frac{\kappa T_\mathrm{h}}{m_\mathrm{i}}. \tag{40}$$

We are admittedly dealing here with a very special case, and would need a complete revision of the expansion scheme, leading to a KdV–ZK equation with quartic nonlinearities. Because the physical insight is limited, we shall not pursue this further, but have only indicated this here because supercritical densities are rarely mentioned correctly in the literature, if at all (Verheest and Hellberg 1999).

5.3. Ion-acoustic solitons in dusty plasmas

In this section we investigate the influence of charged dust on the ion-acoustic mode. The model adopted, as in uncontaminated plasmas, is that of Boltzmann electrons and fluid ions, but where part of the electron charge has been absorbed by the background dust, considered immobile on the ion-acoustic timescales. In this three-species model there is no equilibrium streaming, and it follows from (9) that the phase velocity V of these modes obeys

$$V^2 = \omega_{pi}^2\lambda_\mathrm{De}^2 + v_{T\mathrm{i}}^2 = \frac{N_\mathrm{i}}{N_\mathrm{e}}v_\mathrm{ia}^2 + v_{T\mathrm{i}}^2 \tag{41}$$

where $v_\mathrm{ia}^2 = \kappa T_\mathrm{e}/m_\mathrm{i}$ refers to the normal ion-acoustic velocity in uncontaminated plasmas, and we remind ourselves of the fact that $v_{T\mathrm{i}}^2$ contains a factor γ_i. Ion-acoustic solitons in plasmas without charged dust are always compressive in the electron density, following from (19) because then $B > 0$. We can now ask whether owing to the presence of charged dust, the solitons might also become rarefactive in a given parameter range, in other words, whether B can change sign and become negative.

We thus write the critical density condition $B = 0$, namely

$$\frac{\omega_{pi}^2(3V^2 + v_{Ti}^2)}{m_i(V^2 - v_{Ti}^2)^3} = \frac{1}{\lambda_{De}^2 \kappa T_e} \tag{42}$$

as

$$\left(\frac{N_i}{N_e}\right)^2 - 3\frac{N_i}{N_e} - 4\frac{v_{Ti}^2}{v_{ia}^2} = 0 \tag{43}$$

the positive root of which is approximately given by

$$\frac{N_i}{N_e} \simeq 3 + \frac{4v_{Ti}^2}{3v_{ia}^2}. \tag{44}$$

This is only slightly larger than three because $v_{Ti} \ll v_{ia}$, and hence critical densities occur when $N_i \simeq 3N_e$. The conclusion is that at sufficiently high levels of electron depletion onto the charged dust grains, about two-thirds of the electrons, the solitons become rarefactive ($B < 0$). At or around critical densities ($B \simeq 0$) the KdV–ZK equation is no longer the appropriate paradigm, but the mKdV–ZK equation (24) or mixed KdV–ZK equation (28) has to be used.

5.4. Dust-acoustic solitons

For dust-acoustic solitons the dust motion has to be taken into account, and the standard composition will be taken, that of Boltzmann electrons and ions besides the charged dust. The dispersion law (9) becomes

$$V^2 = \omega_{pd}^2 \lambda_D^2 + v_{Td}^2 = v_{da}^2 + v_{Td}^2 \tag{45}$$

where the dust-acoustic velocity $v_{da} = \omega_{pd}\lambda_D$ has been introduced, and λ_D is now determined by $\lambda_D^{-2} = \lambda_{De}^{-2} + \lambda_{Di}^{-2}$. The expression for B thus becomes

$$B = \frac{q_d(3v_{da}^2 + 4v_{Td}^2)}{m_d v_{da}^4 \lambda_D^2} + \frac{e}{\lambda_{Di}^2 \kappa T_i} - \frac{e}{\lambda_{De}^2 \kappa T_e} \tag{46}$$

which can be rewritten as

$$B = \frac{\varepsilon_0}{N_d q_d}\left[\frac{4v_{Td}^2}{v_{da}^2 \lambda_D^4} + \frac{2}{\lambda_D^4} + \frac{1}{N_e N_i}\left(\frac{N_e}{\lambda_{Di}^2} + \frac{N_i}{\lambda_{De}^2}\right)^2\right]. \tag{47}$$

This is always nonzero and has the sign of q_d, so there are no critical densities. The solitons are compressive or rarefactive, depending on whether the dust is positively or negatively charged, respectively. Similar conclusions are valid for the ordinary KdV equations, when one studies electrostatic waves in unmagnetized dusty plasmas, because the expression for B does not change.

6. Conclusions

To summarize, we have considered the three-dimensional propagation of electrostatic modes oblique to a static magnetic field and used the reductive perturbation approach to derive a generalized KdV–ZK and mKdV–ZK equation governing the nonlinear propagation of these modes. Unlike previous studies that are often restricted to specific physical models and their corresponding normalizations, we have consciously presented unnormalized governing equations that provide deeper physical insight into the inherent nonlinear behaviour dictated by critical densities, temperatures etc, when applied to specific physical models. We obtained a natural bifurcation point, where by assuming that the coefficient of quadratic nonlinearity, B, is non-zero, we obtain the KdV–ZK equation and then the mKdV–ZK equation for when B goes to zero (a condition that is achievable for special plasma compositions), which has inherent cubic nonlinearity. It is apparent from the unnormalized general equations that slow time variations are solely due to the adiabatic fluid constituents of the plasma, whilst oblique effects includes terms entering both through the Laplacian and the fluid species. On the other hand, the hot Boltzmann species affect just the coefficient of the nonlinear term. Further, double layers become possible in the vicinity of critical densities and when both quadratic and cubic nonlinearities are of the same order, such that both terms can be retained, yielding an nonlinear evolution equation with both quadratic and cubic nonlinearities viz the mixed KdV–ZK equation.

For completeness, our generalized nonlinear equations are then applied to specific plasma models. In the case of weakly nonlinear electron-acoustic waves comprising both hot- and cold-electron species and stationary ions we confirm the results of Mace and Hellberg (2001). When applied to ion-acoustic solitons comprising two Boltzmann species of electrons (characterized by different temperatures) and singly charged fluid ions conditions of critical densities and temperatures are obtained, where analogous results have been derived in the literature strictly for the case of parallel propagation. Further, we show that even supercritical densities (where both the coefficients and quadratic and cubic nonlinearities vanish) are possible in this specific model. However, this being a very special case, it may well be necessary to re-visit the expansion scheme, thus leading to quartic nonlinearities. We next apply our general equations to ion-acoustic solitons in a dusty plasma comprising Boltzmann electrons, fluid ions and stationary background charged dust grains, and obtain critical density conditions. Finally we turn our attention to dust-acoustic solitons comprising Boltzmann electrons and ions and fluid charged dust grains. In the case of both KdV–ZK and ordinary KdV solitons we show that the nature of the soliton is determined solely by the sign of the dust charge, with compressive and rarefactive solitons corresponding to positive and negative dust charge, respectively.

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